Total number of printed pages–5

2 PGDE MTH 5(B)

2013

(December)

MATHEMATICS

Paper / Course : 205(B)

(Operator Theory)

Full Marks : 80

Time : 3 hours

The figures in the margin indicate full marks for the questions.

1. Answer ***any four*** questions : 5×4=20

   (a) Prove that a linear operator an a finite dimensional complex normed linear space $X \neq \{0\}$ has at least one eigenvalue.

   (b) Find the eigenvalues and eigenvectors of the matrix A given by

$$A = \begin{bmatrix} 1 & 2 \\ -8 & 11 \end{bmatrix}$$

(c) Show that the eigenvalues of a unitary matrix have absolute value 1.

(d) Give an example of linear operator which has a spectral value distinct from its eigenvalues.

(e) Let $T \in B(X, X)$, X is a Banach space. If $\|T\| < 1$, then prove that $(I - T)^{-1}$ exists as a bounded linear operator on the whole space X and $(I - T)^{-1} = \sum_{i=0}^{\infty} T^i$

Where the Derives on the right converges in the norm on *B(x, x)*.

(f) Show that the resonant set $\int(T)$ of a bounded linear operator T on a Complex Banach space X is open.

2. Answer ***any four*** questions from the following : 4×5=20

(a) What do you mean by a compact linear operator ? Give examples. Show that every compact linear operator is bounded.

(b) Let X be a normal linear space. If dim $X = \infty$ show that the identity operator $I : X \rightarrow X$ is not compact.

(c) Let B be a subset of a metric space X. Then, if B is relatively compact prove that B is totally bounded.

(d) Show that the range $R(T)$ of a compact linear operator $T : X \rightarrow Y$ is separable.

(e) Show that the set of the eigenvalues of a compact linear operator on a normed linear space is countable.

(f) Let $T : X \rightarrow X$ be a compact linear operator on a normed space X. Then show that for every $\lambda \neq 0$, the range of $T_\lambda = T - \lambda I$ is closed.

3. Answer ***any two*** questions from the following : 10×2=20

(a) Let $T : H \rightarrow H$ be a bounded self-adjoint linear operator on a complex Hilbert space *H*. Then prove that,

(i) All eigenvalues of *T* (if they exist) are real.

(ii) Eigenvectors corresponding to numerically different eigenvalues of *T* are orthogonal. 5+5=10

(b) Prove that the residual spectrum $6_r(T)$ of a bounded self-adjoint linear operator $T : H \rightarrow H$ on a complex Hilbert space *H* is empty.

(c) The product *ST* of two bounded self adjoint linear operators *S* and *T* on a Hilbert space *H* is positive if they commute in *ST*=*TS*. Prove this.

4. Answer ***any two*** questions : 2×10=20

(a) Prove that if a linear operator T is defined on all of a complex Hilbert space H and satisfies $\langle T_x, y\rangle = \langle x, Ty\rangle$ for all x, $y \in H$ then T is bounded.

(b) Prove that the Hilbert adjoint operator T^* of a linear operator T is also linear.

(c) Show that $(T_1 T_2)T_3 = T_1(T_2 T_3)$ where T_1, T_2, T_3 are three unbounded linear operators.

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