2012

# Paper : 205-B

## (Operator Theory)

1. Answer any *four* questions : 5×4=20

(a) Show that all matrices representing a given linear operator $T : X \rightarrow X$ on a finite dimensional normed space X relative to various bases for X have the same eigen values.

(b) Show that in a finite dimensional space X, spectrum of any operator is eqaul to its point spectrum.

(c) Prove that for a bounded linear operator on a normed linear space, eigen vectors corresponding to distinct eigen values are linearly independent.

(d) Show that $\rho(T)$ of a bounded linear operator $T : X \rightarrow X$ on a complex Banach space X is compact and lies in the disk given by $|\lambda| \leq \|T\|$.

(e) Define Banach algebra. Show that the set of all continuous complex valued functions defined over [a, b] is a commutative Banach algebra with identity.

(f) Let X be a finite dimensional inner product space and $T : X \to X$ a linear operator. If T is self-adjoint, show that $\sigma(T) \leq \mathbb{R}$ (reals).

2. Answer any *four* questions : 5×4=20

(a) Let X and Y be any two normed linear space and $T : X \to Y$ is linear. Show that T is completely continuous iff for every sequence

$$\{x_n\} \text{ in } X :$$

$$\|x_n\| \leq K \quad \forall_n$$

$\{Tx_n\}$ admits a convergent subsequence.

(b) Define compact operator. Show that any linear operator on a finite dimensional space is compact.

(c) Explain weak and strong convergence in a normed space. Show that strong convergence imply weak convergence. Give an example to show that the converse is not necessarily true.

(d) Let X be a normed space and $\{x_n\} \to x$ weakly in X. If T is a compact operator on X, show that $\{Tx_n\} \to Tx$ strongly.

(e) Show that the projection of a Hilbert space H onto a finite dimensional subspace of H is compact.

(f) Let T be a compact operator on a normed space X. Let $A = T - \lambda I$, $\lambda \neq 0$. Show that N(A) is a finite dimensional subspace of X.

3. Answer any *two* questions : $10 \times 2 = 20$

(a) Let T be a bounded self-adjoint operator on a Hilbert space H. Show that

$\lambda \in \rho(T)$ if $\exists\, c > 0 :$
$\|(T - \lambda I)x\| \geq c\|x\| \quad \forall x \in H$.

(b) Let T be as in Q. 3(a) above.

Let $M = \sup\limits_{\|x\|=1} \langle Tx, x\rangle$ and $m = \inf\limits_{\|x\|=1} \langle Tx, x\rangle$

Show that m, $M \in \sigma T$.

(c) Let T be a bounded linear operator on a Hilbert space H and M a closed subspace of H.

Suppose P is a projection on M. Show that M is reducing under T iff TP = PT.

(d) Let P and Q be projections on a Hilbert space H with ranges M and N respectively. Show that

$Q \leq P$ iff $PQ = QP = Q$.

4. Answer any *two* questions : 2×10=20

(a) Define symmetric linear operator. Show that a linear operator T is symmetric iff T is densely defined and $T < T^*$.

(b) Let X and Y be normed spaces and $A : D(A) \subseteq X \to Y$ is a linear transformation. Show that A is closed iff $G_A$ (the graph of A) is a closed linear subspace of $X * Y$.

(c) Let H be a Hilbert space. $A : D(A) \subseteq H \to H$ and $B : D(B) \subseteq H \to H$ be unbounded linear transformations such that $\overline{D(A)} = H = \overline{D(B)}$.

Show that

(i) $(\alpha A)^* = \bar{\alpha} A^*$

(ii) $A < B \Rightarrow B^* < A^*$

(iii) $(A + \alpha I)^* = A^* + \bar{\alpha} I$

(d) Let H be a Hilbert space and $A : D(A) \subseteq H \to H$ is a densely defined linear transformation permitting closure. Show that

(i) $(\overline{A})^* = A^*$

(ii) $\overline{A} = A^{**}$

(iii) $A = A^{**}$ if A is closed.