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2 PGDE MTH 5(A)

2013

(December)

MATHEMATICS

Paper : 205 (A)

(Number Theory)

Full Marks : 80

Time : Three hours

The figures in the margin indicate full marks for the questions.

1. Answer *any four* questions. 4×5=20

(a) Let p be an odd prime and $gcd\ (a, p) = 1$. Then show that a is a quadratic residue of p if and only if $a^{\frac{p-1}{x}} \equiv 1 \pmod{p}$

(b) If p is an odd prime, then prove that

$$\left(\frac{2}{p}\right) = \begin{cases} 1, \text{ if } p \equiv 1 \pmod{8} \text{ or } \\ \qquad p \equiv 7 \pmod{8} \\ -1, \text{ if } p \equiv 3 \pmod{8} \text{ or } \\ \qquad p \equiv 5 \pmod{8} \end{cases}$$

(c) Determine whether the following quadratic congruence is solvable :

$$3x^2 + 6x + 5 \equiv 0 \pmod{89}$$

(d) What do you mean by a primitive root of n? Show that 3 is a primitive root of 17. Hence, determine all primitive roots of 17.

(e) If p and q are distinct odd primes, then show that

$$\left(\frac{p}{q}\right) = \begin{cases} \left(\frac{q}{p}\right) \text{ if } p \equiv 1 \pmod 4 \text{ or } \\ \quad q \equiv 1 \pmod 4 \\ -\left(\frac{q}{p}\right) \text{ if } p \equiv q \equiv 3 \pmod 4 \end{cases}$$

Hence evaluate $\left(\frac{29}{53}\right)$

2. Answer *any four* questions : 4×4=16

(a) Determine all solutions in the positive integers of the Diophantine equation $18x + 5y = 48$.

(b) Let (x, y, z) be a primitive Pythagorean Triplet. Show that one of the integers x or y is divisible by 3 and one of the integers x, y or z is divisible by 5.

(c) Prove that a prime of the form $4k+1$ can be expressed as a sum of two squares.

(d) Prove that no positive integer of the form $4^n(8k+7)$ can be expressed as the sum of three squares.

(e) Given an odd prime p, show that there exists an integer $k < p$ such that kp is the sum of four squares.

3. Answer ***any four*** questions : $4\times4=16$

(a) Using the notion of Farey sequence establish the following :

If $(a, b)=1$ then the equation $ax+by=1$ is solvable.

(b) For an irrational number γ and a positive integer N, show that there always exists a rational number $\frac{h}{k}$ with $k \leq N$ such that $\left|\gamma-\frac{h}{K}\right|<\frac{1}{(N+1)k}$.

(c) (i) Express $\frac{19}{51}$ as a finite simple continued fraction.

(ii) Determine the unique irrational number represented by the infinite continued fraction $[\overline{1;4}]$

(d) If $C_k = p_k / q_k$ is the kth convergent of the simple continued fraction $\lfloor a_0 ; a, a_2, \cdots a_n \rfloor$ then prove that

$$p_k q_{k-1} - q_k p_{k-1} = (-1)^{k-1}, \quad 1 \le k \le n$$

(e) Prove that the greatest common divisor of two Fibonacci numbers is itself a Fibonacci number.

(f) Solve : $x^2 - 2qy^2 = 1$

4. Answer ***any two*** questions : 2×8=16

(a) (i) If α and β are algebraic numbers, show that $\alpha + \beta$ and $\alpha\beta$ are also algebraic numbers.

(ii) If α is any algebraic number, show that there is a rational integer b such that $b\alpha$ is an algebraic integer.

(b) Establish the following :

(i) the norm of a product equals the product of the norms.

(ii) the norm of an integer in $Q\left(\sqrt{m}\right)$ is a rational integer.

(iii) if γ is an integer in $Q\left(\sqrt{m}\right)$ then the norm $N\left(\gamma\right)=\pm1$ if and only if γ is a unit.

(c) Prove that an imaginary quadratic field has only a finite number of units. Further show that every real quadratic field has infinitely many units.

5. Answer *any two* questions : $2\times6=12$

(a) Prove that the fields $Q\left(\sqrt{m}\right)$ for $m=-3$ and -7 have the unique factorization property.

(b) Let $Q(\sqrt{m})$ have the unique factorization property. If $(2, m) = 1$, then prove the following :

(i) 2 is the associate of a square of a prime if $m = 3 \pmod 4$

(ii) 2 is a prime if $m = 5 \pmod 8$

(iii) 2 is the product of two distinct primes if $m = 1 \pmod 8$

(c) Let $P_m(n)$ denote the number of partitions of n into summands less than or equal to m.

If $0 \le x \le 1$ and $\phi_n(x) = \prod_{n=1}^{m} (1 - x^n)$ then

prove that $\sum_{n=0}^{\infty} P_m(n) x^n$ converges

and $\sum_{n=0}^{\infty} P_m(n) x^n = \dfrac{1}{\phi_n(x)}$