2012

(Nov.-Dec.)

MATHEMATICS

Paper : 205 (A/B/C)

Full Marks – 80

Time – Three hours

The figures in the margin indicate full marks for the questions.

Paper : 205-A

(Number Theory)

Answer any *two* questions. 2×10=20

1. (a) Let p be an odd prime. Then establish the following :

(i) $\left(\frac{a}{p}\right) \equiv a^{\frac{p-1}{2}} \pmod{p}$, if $(a, p) = 1$.

(ii) $\sum_{a=1}^{p-1} \left(\frac{a}{p}\right) = 0$ 2+3=5

(b) Let p be an odd prime. Then prove that $x^2 \equiv 2 \pmod{p}$ has solutions if and only if $p \equiv 1$ or 7 (mod 8). 5

2. (a) Determine the quadratic residues of 17. 3

(b) State the Quadratic Reciprocity Law. Hence evaluate $\left(\frac{-23}{59}\right)$. 1+3=4

(c) Let p be an odd prime and (a, p) = 1. If $p \equiv 1 \pmod{4}$, show that p – a is a quadratic residue of p if and only if *a* is a quadratic residue of p. 3

3. (a) Find all primes p for which –3 is a quadratic non-residue. 3

(b) If Q is an odd integer and Q > 0, show that

$$\left(\frac{-1}{Q}\right) = (-1)^{\frac{Q-1}{2}}$$ 3

(c) Show that the congruence $x^2 \equiv 15 \pmod{1093}$ has no solution. 4

Answer any *two* questions. 2×8=16

4. (a) Prove that the equation $ax + by = c$ has a solution if and only if d/c where $d = (a, b)$. If x_o, y_o is any particular solution of the equation, show that all other solutions are given by

$$x = x_o + \left(\frac{b}{d}\right)t$$

$$y = y_o - \left(\frac{a}{d}\right)t$$ for varying integers z. 6

(b) Show that no prime p of the form 4k+3 is a sum of two squares. 2

5. (a) Prove that the equation $x^4 + y^4 = z^2$ has no positive integral solution. 5

(b) If $x^2 + y^2 = z^2$, prove that one of x, y is congruent to $\pm 1 \pmod 4$ and the other is congruent to $0 \pmod 4$. 3

6. (a) Let p denote any odd prime. Show that there is an integer m such that $1 \leq m < p$ and $mp = x_1^2 + x_2^2 + x_3^2 + x_4^2$ for some integers x_1, x_2, x_3, x_4. 5

(b) If p is an odd prime, show that there exist integers a, b such that $a^2 + b^2 \equiv -1 \pmod{p}$. 3

Answer any *two* questions. 2×8=16

7. (a) Show that no two consecutive terms of F_n, a Farey sequence of order n have same denominator. 3

(b) Determine the rational number represented by the simple continued fraction [0, 2, 1, 2, 6]. Also determine the convergents of the fraction. 3+2=5

8. (a) Prove that the value of any infinite continued fraction is an irrational number. 5

(b) If F_n denotes the *n*th term of the Fibonacci sequence, show that

$$F^2_{n+1} - F_n F_{n+2} = (-1)^n$$ 3

9. Find a solution of the equation $x^2 - 21y^2 = 1$. 8

Answer any *two* questions. $2 \times 8 = 16$

10. (a) Prove that an algebraic number ξ satisfies a unique irreducible monic polynomial equation $g(x) = 0$. Further show that every polynomial equation over Q satisfied by ξ is divisible by $g(x)$. 5

(b) Prove that the integers of any algebraic number field form a ring. 3

11. (a) What do you mean by the term 'Norm' of a number α in $Q\left(\sqrt{m}\right)$. Prove that the norm of a product equals the product of the norms. 3

(b) If α, β are integers in $Q\left(\sqrt{m}\right)$ and if $\alpha | \beta$, show that $N(\alpha) | N(\beta)$. (with usual notations) 2

(c) If the norm of an integer α in $Q\left(\sqrt{m}\right)$ is $\pm p$, where p is a rational prime. Show that α is a prime. 3

12. Establish the following :
Every quadratic field is of the form $Q(\sqrt{m})$ where m is a square-free rational integer, positive or negative but not equal to 1. Numbers of the form $a + b\sqrt{m}$ with rational integers a and b are integers of $Q(\sqrt{m})$. These are the only integers of $Q(\sqrt{m})$ if $m \equiv 2$ or $3 \pmod 4$. If $m \equiv 1 \pmod 4$, the numbers $(a + b\sqrt{m})/2$ with odd rational integers a and b are also integers of $Q(\sqrt{m})$ and there are no further integers. 8

Answer any *two* questions. 2×6=12

13. Prove that the fields $Q(\sqrt{m})$ for $m = -1, -2, 2, 3$ are Euclidean and so have the unique factorization property. 6

14. Let $Q(\sqrt{m})$ have the unique factorization property. Prove that an odd rational prime p satisfying $(p, m) = 1$ is a product $\pi_1 \pi_2$ of two primes in $Q(\sqrt{m})$ if and only if $\left(\frac{m}{p}\right) = 1$. 6

15. Prove the following : 6

For $n \geq 1$, the number of partitions of n into distinct summands is the same as the number of partitions of n into odd summands. 6