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2 PGDE MTH 1

2012

(Nov.-Dec.)

MATHEMATICS

Paper : 201

(Topology and Functional Analysis)

Full Marks – 80

Time – Three hours

The figures in the margin indicate full marks for the questions.

(Topology)

Marks – 40

1. Answer any *two* questions : 2×6=12

(a) (i) Define a topology on a non-empty set. 1

(ii) Give an example of a topological space. 1

(iii) Define a base for a given topology. 1

(iv) Define subbase for a given topology. 1

(v) Define a neighbourhood of a point in a topological space. 1

(vi) Define the closure of a subset of a topological space. 1

(b) Let X and Y be topological spaces and $f : X \to Y$. Prove that f is continuous on X if and only if the inverse image of every basic open set of Y is open in X.

(c) Show that if A is closed in X and B is closed in Y, then $A \times B$ is closed in $X \times Y$. Where X, Y are topological spaces.

(d) Suppose that $f : X \to Y$ is continuous. If x is a limit point of a subset A of X, is it necessary that f(x) is a limit point of f(A) ? Justify your answer. Here X, Y are topological spaces.

2. Answer any *two* questions : $2 \times 6 = 12$

(a) (i) Define a T_1 - topological space. 1

(ii) Define a Hausdorff topological space. 1

(iii) Define a regular topological space. 1

(iv) Define a completely regular topological space. 1

(v) Define a T_3 - topological space. 1

(vi) Define a normal topological space. 1

(b) Prove that a topological space is T_1 if and only if each singleton subset is closed.

(c) Let X, Y be topological spaces and f, g : $X \to Y$ be two continuous functions. If Y is a Hausdorff space, then prove that $A = \{x : f(x) = g(x)\}$ is a closed subset of X.

(d) Prove that a completely regular space is regular.

3. Answer any *two* questions : 2×4=8

(a) Prove that any closed subspace of a compact space is compact.

(b) If E is an infinite subset of a compact space X, then prove that E has a limit point in X.

(c) Let $f : X \to Y$ be a bijective and continuous function. If X is compact and Y is Hausdorff, then prove that f is a homomorphism.

(d) Let X be a Hausdorff space and A be a subset of X. If A is compact, then prove that A is closed.

4. Answer any *two* questions : 2×4=8

(a) Let X be a connected topological space. Prove that the only subsets which are both open and closed are ϕ and X.

(b) Let Y be a subspace of a topological space X and A, B $\subseteq$ Y. If A and B are separated in X, then prove that they are separated in Y.

(c) Prove that the union of a collection of connected sets that have a point in common is connected.

(d) Prove that a path connected space is connected

(Functional Analysis)

Marks – 40

1. Answer any *one* question : 8

(a) Define a normed linear space and a Banach space. Prove that every finite dimensional subspace of a normed linear space is a Banach space. 1+1+6=8

(b) Let $(X, \|.\|)$ be a normed linear space and M a closed subspace of X. Define a mapping $\|.\|_q : X|M \rightarrow R$, given by $\|x + M\|_q = \inf \{\|x + m\| : x + m \in x + M\}$ for all $x + M \in X|M$. Then show that $\|.\|_q$ is a norm on X|M. 8

2. Answer any *one* question : 8

(a) Let X and Y be normed linear spaces and $T : X \rightarrow Y$ a linear operator. Prove that T is continuous on X if and only if it is continuous at any point in X.

(b) State open mapping theorem. State and prove closed graph theorem. 1+7=8

3. Answer any *one* question : 8

(a) Define a bounded linear functional on a normed linear space. Prove that a linear functional on a normed linear space is continuous if its kernel is closed. 1+7=8

(b) State and prove the generalized Liouville's theorem.

4. Answer any *one* question : 8

(a) Define an inner product space. State and prove Schwartz inequality. 1+7=8

(b) Define a Hilbert space. Let H be a Hilbert space. If M and N are closed subspaces of H, then prove that M+N is a closed subspace of H. 8

5. Answer any *one* question : 8

(a) Let H_1 and H_2 be Hilbert spaces and $T : H_1 \to H_2$ be a bounded linear operator. Then prove that there exists a unique bounded linear operator $T^* : H_2 \to H_1$ such that $\langle T(x), y \rangle = \langle x, T^*(y) \rangle \ \forall x \in H_1$ and $\forall y \in H_2$. 8

(b) Define a normal operator, a unitary operator and a projection operator.

Let X be a Banach space, and M and N be closed linear subspaces of X such that $X = M \oplus N$. Then prove that there exists a unique linear projection P on X whose range and null spaces are M and N respectively. 1+1+1+5=8